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# Linear operators and c.o.n.s. in Hilbert spaces associated with chaotic dynamical systems

## (カオス力学系に付随する線形作用素および完全正規直交系)

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In this note, we will show a non-chaotic property in chaotic dynamical system $(X, \varphi)$, where $X$ is a compact set and $\varphi$ is a continuous map of $X$ onto $X$ . One of the most important property in the chaotic dynamical theory is to be sensitive dependence on initial conditions, whichi is a chaotic property in the sense that there exists $\delta > 0$ such that, for any $x$ in $X$ and neighbourhood $U(x)$ of $x$, there exists $y$ in $U(x)$ and $n \geq 0$ such that

$$d(\varphi^n(x), \varphi^n(y)) > \delta.$$

On the other hand, some chaotic dynamical systems have the property of toplogical-mixing on a measure spoce $(X, m)$, that is,

$$\lim_{n\to\infty} \int_X f(\varphi^n(x))g(x)dm = \int_X f(x)dm$$

for any continuous function $f$ on a metric space $X$ and $L^1$- function $g$ on the measurable space $(X, m)$ with $\int_X g(x)dm = 1$.

We study this non-chaotic property by representing chaotic dynamical systems $(X, \varphi)$ on Hilbert spaces $\mathfrak{H}$.

### 1. Covariant representation of dynamical systems

Let $C(X)$ be the $C^*$-algebra of all continuous functions on $X$. Then a continuous map $\varphi$ from $X$ onto itself induces a $*$-endomorphism $\alpha_\varphi$ of $C(X)$, which is defined by

$$\alpha_\varphi(f)(x) = f(\varphi(x)), \quad x \in X.$$

Let $\pi$ be a covariant representation of $(C(X), \alpha_\varphi)$ on a Hilbert space $\mathfrak{H}$ in the following sense:

$$\pi(\alpha_\varphi(f)) = V_1\pi(f)V_1^* + V_2\pi(f)V_2^*$$

for all $f$ in $C(X)$, where $(V_1, V_2)$ is a couple of isometries on $\mathfrak{H}$ with the property

$$V_1V_1^* + V_2V_2^* = I.$$

In this case, $(V_1, V_2)$ induces a $*$-endomorphism of $\mathfrak{L}(\mathfrak{H})$ as follows:

$$\alpha_V(a) = V_1 a V_1^* + V_2 a V_2^*$$

for all $a$ in $\mathfrak{L}(\mathfrak{H})$.

For some couples $(V_1, V_2)$, we can find a c.o.n.s. $\{e_n\}_{n=1}^{\infty}$ satisfying following condition:

$$V_1 e_n = e_{2n-1} \quad \text{and} \quad V_2 e_n = e_{2n} \quad \text{for all } n \geq 1,$$

which are called a c.o.n.s. of Walsh type with respect to $(V_1, V_2)$.

Related to these systems, we have already had a theorem which shows a non-chaotic property.

**Theorem 1.1.** [2:Theorem 2.2.3] *Let $\pi$ be a covariant representation of $(C(X), \alpha_\varphi)$ implememted by $(V_1, V_2)$. If $(V_1, V_2)$ has a c.o.n.s. $\{e_n\}_{n=1}^{\infty}$ of Walsh type, then we have*

$$\lim_{n\to\infty} (\alpha_V^n(a)\xi, \xi) = (ae_1, e_1)$$

*for all $a$ in $A$ and $\xi$ in $\mathfrak{H}$ with $\| \xi \| = 1$.*

Here we give some examples. Let $\varphi$ be a unimodal map of $[0,1]$ onto itself in the following sense.

(1) $\varphi$ is a continuous map of $[0,1]$ onto $[0,1]$.

(2) There exists a point $c$ in $(0,1)$ such that

 (i) $\varphi(0) = \varphi(1) = 0$ and $\varphi(c) = 1$,

 (ii) $\varphi$ is strictly monotone increasing on $[0,c]$ and strictry monotone decreasing on $[c,1]$,

 (iii) $\varphi$ and the two inverse maps $\beta, \gamma$ of $\varphi$ are abosolutely continuous functions on $[0,1]$, where $\beta([0,1]) = [0,c]$ and $\gamma([0,1]) = [c,1]$.

Given a unimodal map $\varphi$, we define a couple $(V_1, V_2)$ of isometries associated with $\varphi$ as follows:

$$V_1 = V_1(\varphi) = M_{\sqrt{\varphi'}} M_{\chi_{[0,c]}} T_\varphi \quad \text{and} \quad V_2 = V_2(\varphi) = -M_{\sqrt{-\varphi'}} M_{\chi_{[c,1]}} T_\varphi,$$

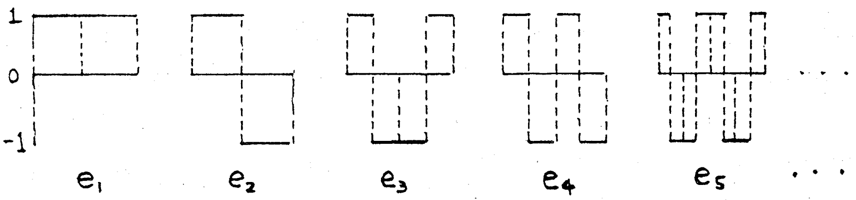
where $M_f$ means the multiplication operator on $L^2[0,1]$, $\chi_E$ the characteristic function of $E$ and $(T_\varphi \xi)(x) = \xi(\varphi(x))$.

Let $\pi(f) = M_f$ for $f$ in $C(X)$. Then $\pi$ is a covariant representation of $(C(X), \alpha_\varphi)$ with respect to this couple $(V_1, V_2)$, but it has no c.o.n.s. of Walsh type. However, in the following example, we can find a couple $(W_1, W_2)$ of isometris which implements the $*$-endomorphism $\alpha_V$ and has a c.o.n.s of Walsh type.

**Example 1.2.** Let $X = [0,1]$ and $\varphi = 1- \mid 1 - 2x \mid$ : the tent map.

$$W_1 = \frac{1}{\sqrt{2}}V_1 - \frac{1}{\sqrt{2}}V_2 (= T_\tau) \quad \text{and} \quad W_2 = W_2(\tau) = \frac{1}{\sqrt{2}}V_1 + \frac{1}{\sqrt{2}}V_2.$$

Then $\pi$ is also a covariant representation of $(C([0,1]), \alpha_\varphi)$ implemented by $(W_1, W_2)$ having a c.o.n.s. $\{e_n\}_{n=1}^{\infty}$ of Walsh type, which is just the following Walsh series (cf.[3]).




Putting $a = M_f$ for $f$ in $C(X)$ in the theorem above, we have

$$\lim_{n \to \infty} (\alpha_\varphi^n(f)\xi, \xi) = \lim_{n \to \infty} (\alpha_V^n(f)\xi, \xi) = (fe_1, e_1) = (f, e_1)$$

for all $\xi$ in $\mathfrak{H}$ with $\| \xi \| = 1$.

**Remark.1.3.** The Walsh series becomes a group with respect to product of fuctoins on [0,1], which satisfies the following relation.

Group $G = \{e_n\}_{n=1}^{\infty}$

| | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ | $e_8$ | $\cdots$ |
|---|---|---|---|---|---|---|---|---|---|
| $e_1$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ | $e_8$ | |
| $e_2$ | $e_2$ | $e_1$ | $e_4$ | $e_3$ | $e_6$ | $e_5$ | $e_8$ | $e_7$ | |
| $e_3$ | $e_3$ | $e_4$ | $e_1$ | $e_2$ | $e_7$ | $e_8$ | $e_5$ | $e_6$ | |
| $e_4$ | $e_4$ | $e_3$ | $e_2$ | $e_1$ | $e_8$ | $e_7$ | $e_6$ | $e_5$ | |
| $e_5$ | $e_5$ | $e_6$ | $e_7$ | $e_8$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | |
| $e_6$ | $e_6$ | $e_5$ | $e_8$ | $e_7$ | $e_2$ | $e_1$ | $e_3$ | $e_4$ | |
| $e_7$ | $e_7$ | $e_8$ | $e_5$ | $e_6$ | $e_3$ | $e_4$ | $e_1$ | $e_2$ | |
| $e_8$ | $e_8$ | $e_7$ | $e_6$ | $e_5$ | $e_4$ | $e_3$ | $e_2$ | $e_1$ | |
| $\vdots$ | | | | | | | | | |

Let $\varphi$ be topologically conjugate to the tent map $\tau$, that is, $\varphi = h \circ \tau \circ h^{-1}$ for some homeomorphism $h$ of [0,1] onto itself. In our case, the maps $h$ and $h^{-1}$ are assumed to be absolutely continuous functions on [0,1].

Then $(C(X), \alpha_\varphi)$ has a covariant representation implemented by $(W_1, W_2)$ defined as in Example 1.2. The couple $(W_1, W_2)$ has a c.o.n.s. $\{e_n\}_{n=1}^{\infty}$ of Walsh type with $e_1 = \sqrt{(h^{-1})'}$, where $(h^{-1})'$ is the derivative of $h^{-1}$.

**Example 1.4.** Let $X = [0,1]$ and $\varphi = 4x(1-x)$ : the logistic map. Then $\varphi$ is topologically cojugate to the tent map with conjugacy $h(x) = \sin^2(\pi x/2)$. Thus the couple $(W_1, W_2)$ has a c.o.n.s. $\{e_n\}_{n=1}^{\infty}$ of Walsh type with $e_1(x) = 1/(\pi(x(1-x))^{1/2})^{1/2}$.

**2. Convergence of sequences $\{\ (\alpha_V^n(\cdot)\xi, \xi)\ \}_{n=1}^{\infty}$ in $\sigma$-weak topology**

We consider the convergence in Theorem 1.1 in the context of duality between $\mathfrak{L}(\mathfrak{H})$ and the predual space $\mathfrak{L}(\mathfrak{H})_*$. Let $M$ be an $\alpha_V$-invariant von Neuman subalgebra of $\mathfrak{L}(\mathfrak{H})$ and $A = A_V^M$ the adjoint operator of the restriction of $\alpha_V$ to $M$. Then Theorem 1.1 implies the following. If $(V_1, V_2)$ has a c.o.n.s. $\{e_n\}_{n=1}^{\infty}$ of Walsh type, then we have

$$\lim_{n\to\infty} A^n(\omega_{\xi,\xi}) = \omega_{e_1,e_1}$$

for all $\xi$ in $\mathfrak{H}$ with $\| \xi \| = 1$, where $\omega_{\xi,\xi}$ is a vector state on $\mathfrak{L}(\mathfrak{H})$.

Let $M$ be the abelian von Neumann subalgebra $M_{L^\infty[0,1]}$. Then the predual $M_*$ is regarded as $L^1[0,1]$. Let $(W_1, W_2)$ be as in Example 1.2 or 1.4. Then we have $A = A_V^M = A_W^M$ and the convergence mentioned above means the following.

If $\varphi$ is the tent map on [0,1], we have

$$\lim_{n\to\infty} A^n(\eta) = e_1^2 = e_1$$

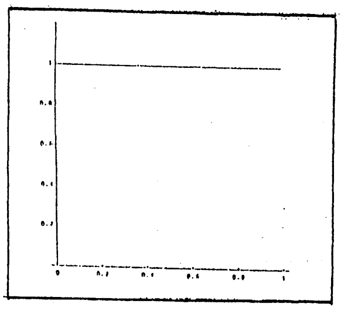
for all $\eta$ in $L^1[0,1]$ with $\| \eta \|_1 = 1$, where $e_1(x) = 1$.

On the other hand, if $\varphi$ is the logistic map on [0,1], we have.

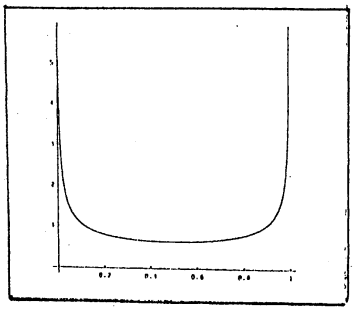
$$\lim_{n\to\infty} A^n(\eta) = e_1^2$$

for all $\eta$ in $L^1[0,1]$ with $\| \eta \|_1 = 1$, where $e_1^2(x) = 1/\pi\sqrt{x(1-x)}$.

$e_1^2(x) = 1$



$e_1^2(x) = \dfrac{1}{\pi\sqrt{x(1-x)}}$



In the case of the tent map, we have more detailed results mentioned below.

**Theorem 2.1** *Let $\varphi$ be the tent map on* [0,1] *and $M$ the abelian von Neumann algebra $M_{L^\infty[0,1]}$. Put $A = A_V^M$. Then we have the following.*

(1) $A(\eta)(x) = \frac{1}{2}(\eta(\frac{x}{2}) + \eta(1-\frac{x}{2}))$ *for $\eta$ in $L^1[0,1]$.*

(2) $\lim_{n\to\infty} A^n(\eta) = 1$ *in $\sigma(L^1[0,1], L^\infty[0,1])$ topology, for $\eta$ in $L^1[0,1]$ with $\| \eta \|_1 = 1$.*

(3) $\lim_{n\to\infty} \| A^n(\eta) - 1 \|_1 = 0$ *for $\eta$ in $L^2[0,1]$ with $\| \eta \|_1 = 1$.*

(4) $\lim_{n\to\infty} \| A^n(\eta) - 1 \|_\infty = 0$ *for $\eta$ in $C_W[0,1]$, where $C_W[0,1]$ is the $C^*$-subalgebra of $L^\infty[0,1]$ generated by the Walish series $\{e_n\}_{n=1}^\infty$.*

We note that

$$C[0,1] \subset C_W[0,1] \subset L^\infty[0,1] \subset L^2[0,1] \subset L^1[0,1].$$

**Remark 2.2.** Let $A$ be the map of $L^1[0,1]$ into $L^1[0,1]$ mentioned in the theorem above. Then we have the following.

(1)$A(1) = 1$.

(2)$A(2x) = 1$.

(3)$A^n(3x^2) = \frac{3}{4^n} - \frac{3}{4^{n-1}\cdot 2} + \frac{4^{n-1}\cdot 2 + 1}{4^{n-1}\cdot 2}$ for each positive integer $n$.

For other $\alpha_V$-invariant von Neuman subalgebras $M$, we have some results concerning the property of convergence of the sequence $\{ (A_V^M)^n(\omega_{\xi,\xi}) \}_{n=1}^\infty$. Moreover we are studying representations of chaotic dynamical systems in the case of other unimodal maps and Dai and Larson [1] provide an interesting theory which is deeply related to our study.

## References


[1] X. Dai and D. Larson, *Wandering vectors for unitary systems and orthogonal wavelets* to appear as Memoire of A.M.S.

[2] S. Kawamura, Covarinat representations chaotic dynamical systems, to appear in Tokyo J. Math.

[3] F. Schipp, W.R. Wade and P. Simon, *Walsh series,* Adam Hilger, Bristol-New York, 1990.